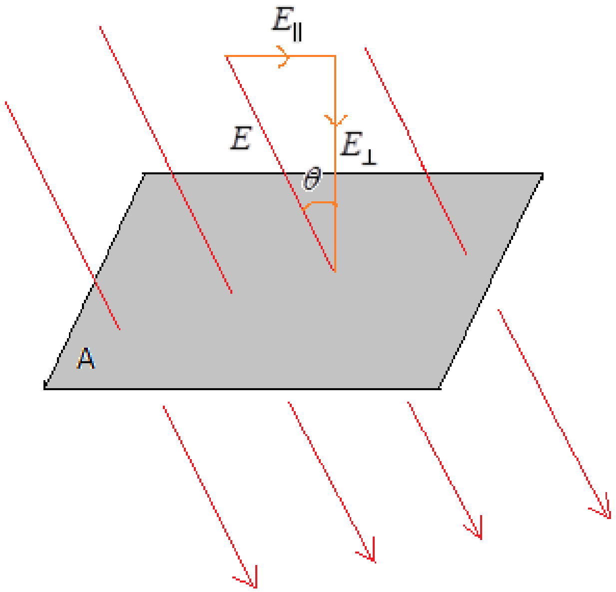
A.2 Gauss's Law

Gauss's law describes a property of electric fields which can be used to calculate these fields much more easily than we can with Coulomb's law, at least in cases of particularly high symmetry. To formulate this law, we first have to introduce the concept of electric flux.



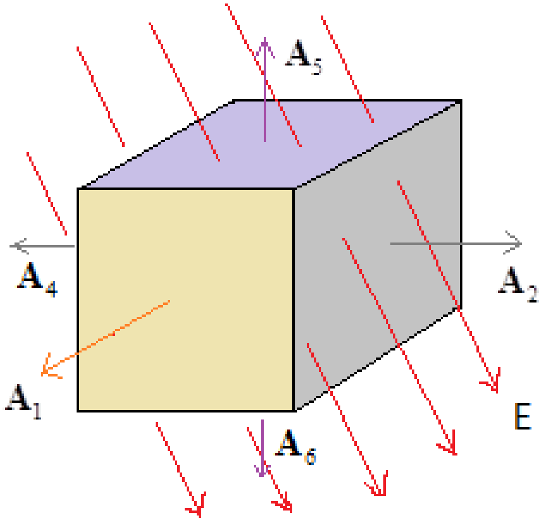
$$\begin{aligned}\Phi &\sim \#\text{ of electric field lines passing through a surface} \qquad [\text{units} = \mathrm{Nm}^2/\mathrm{C}]\\ &= E_{\perp} A\\ &= EA\cos\theta\\ &= \mathbf{E}\cdot\mathbf{A} \qquad \text{where } \mathbf{A} = A \text{ @ direction} \perp \text{surface}\end{aligned}$$

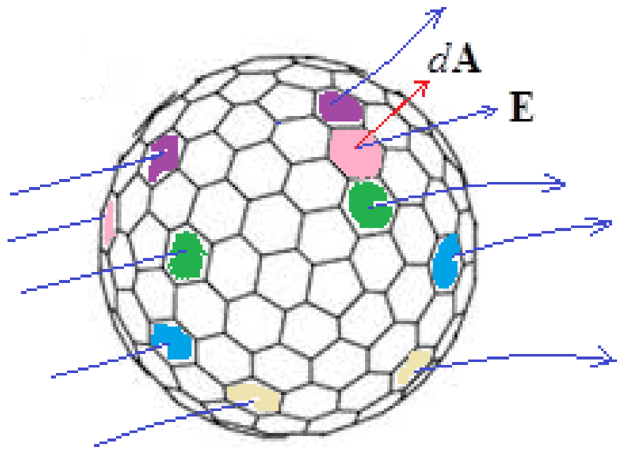
A.2 Gauss's Law

We're (or is it just me?) *particularly* interested in the net flux that passes through a *closed* surface, i.e. a surface that has a definite inside and outside, like this cube to the left, or soccer ball to the right. In this case the net flux would be:

$$\Phi = \sum_{\text{all sides}} \mathbf{E} \cdot \mathbf{A}$$

$$\Phi = \sum_{\text{all sides}} \mathbf{E} \cdot d\mathbf{A} \rightarrow \oint \mathbf{E} \cdot d\mathbf{A}$$

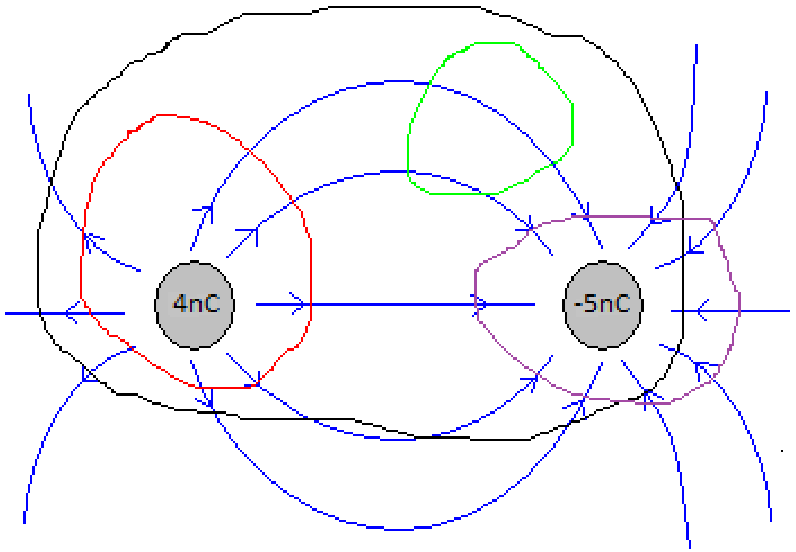




Note that if the electric field lines are going into the surface, they contribute negative flux, and if they leave the surface, they contribute positive flux. And so we can interpret the net flux through a closed surface as being the *net* number of field lines leaving the surface.

A.2 Gauss's Law

We can relate the net flux passing through a closed surface to the charge contained within it.



$$\Phi_{green\,surface} \sim ? \qquad \sim 0\,\mathrm{Nm^2/C^2}$$

$$\Phi_{red\,surface} \sim ? \qquad \sim 8\,\mathrm{Nm^2/C^2}$$

$$\Phi_{purple\,surface} \sim ? \qquad \sim -10\,\mathrm{Nm^2/C^2}$$

$$\Phi_{black\,surface} \sim ? \qquad -2\,\mathrm{Nm^2/C^2}$$

So the net flux passing through a closed surface we see to be proportional to the net charge enclosed within it.
This is essentially Gauss's law:

$$\oint \mathbf{E}\cdot d\mathbf{A} = \frac{q_{enclosed}}{\varepsilon_0} \qquad \text{Gauss's law}$$

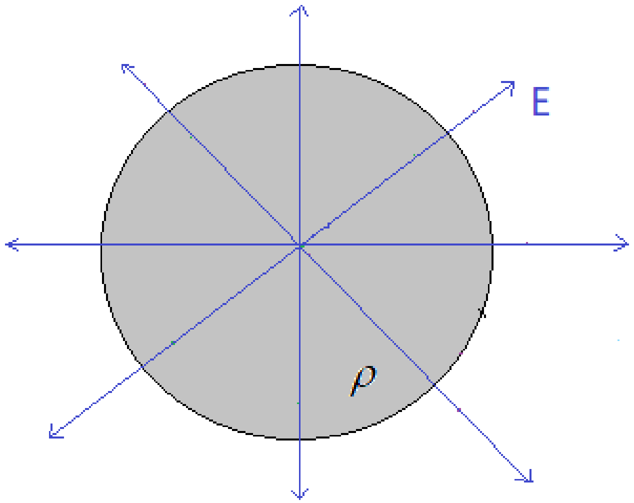
$$\varepsilon_0 = \text{'permittivity of free space'}$$

$$= \frac{1}{4\pi k} = \frac{1}{4\pi(9\times10^9\,\mathrm{Nm^2/C^2})} = 8.85\times10^{-12}\,\frac{\mathrm{C^2}}{\mathrm{Nm^2}}$$

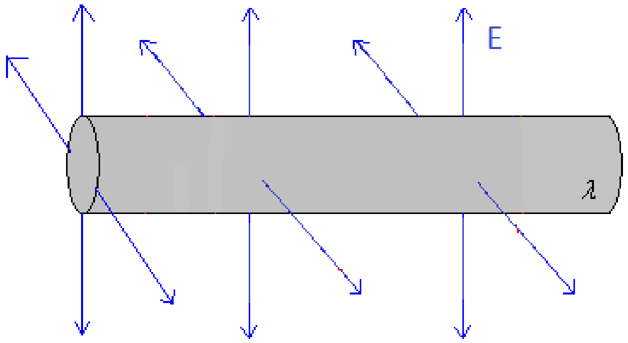
A.2 Hey! Gauss's Law what is it good for? absolutely *something*

Gauss's law is always *true*. But it is primarily *useful*, in the context of calculating **E**, in cases of high charge symmetry:

Spherical Symmetry

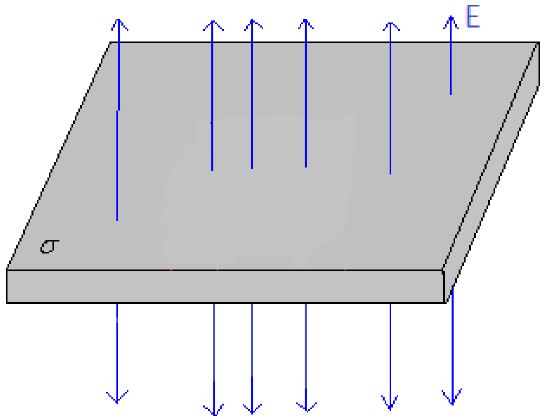


$$\rho = \frac{\text{charge}}{\text{volume}} \qquad [\text{units} = \text{C/m}^3]$$



Cylindrical Symmetry

$$\lambda = \frac{\text{charge}}{\text{length}} \qquad [\text{units} = \text{C/m}]$$

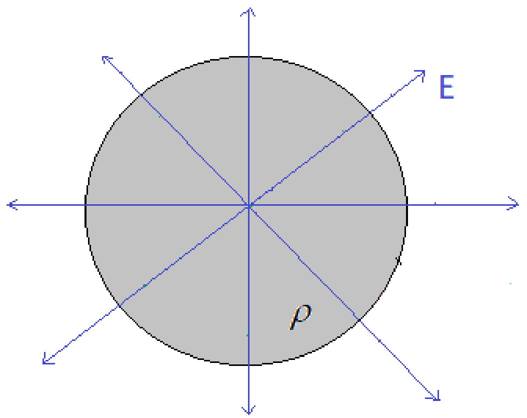


Planar Symmetry

$$\sigma = \frac{\text{charge}}{\text{area}} \qquad [\text{units} = \text{C/m}^2]$$

A.2 Gauss's Law

Now we're going to apply Gauss's law to get the electric field of a spherically symmetric charge distribution.

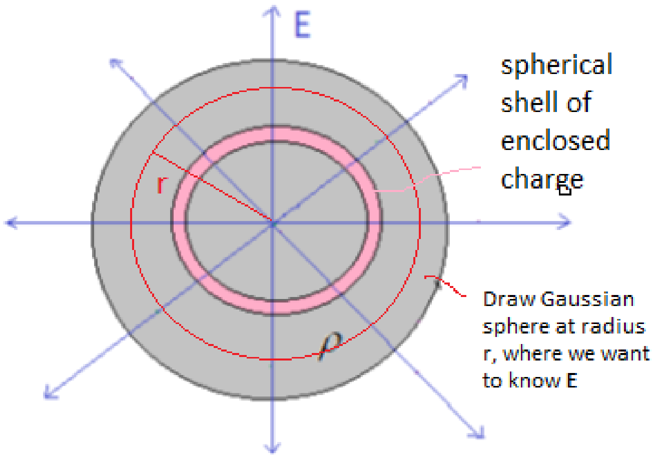


$$\oint \mathbf{E}\cdot d\mathbf{A} = \frac{q_{enc}}{\varepsilon_0}$$

$$\oint E dA = \frac{q_{enc.}}{\varepsilon_0}$$ because $\mathbf{E}$ is parallel to $\mathbf{A}$ everywhere

$$E\oint dA = \frac{q_{enc.}}{\varepsilon_0}$$ E is constant everywhere on surface, so can pull out of integral

$$E 4\pi r^2 = \frac{q_{enc.}}{\varepsilon_0}$$ sum of A over all surfaces just gives total surface area of sphere

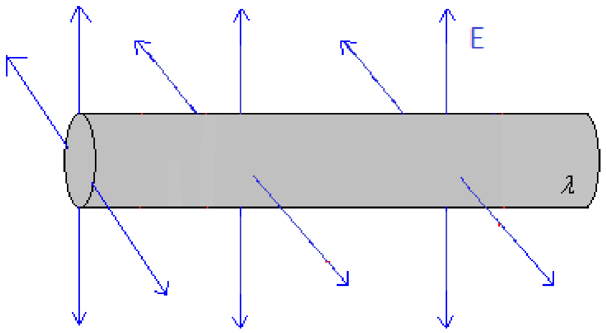


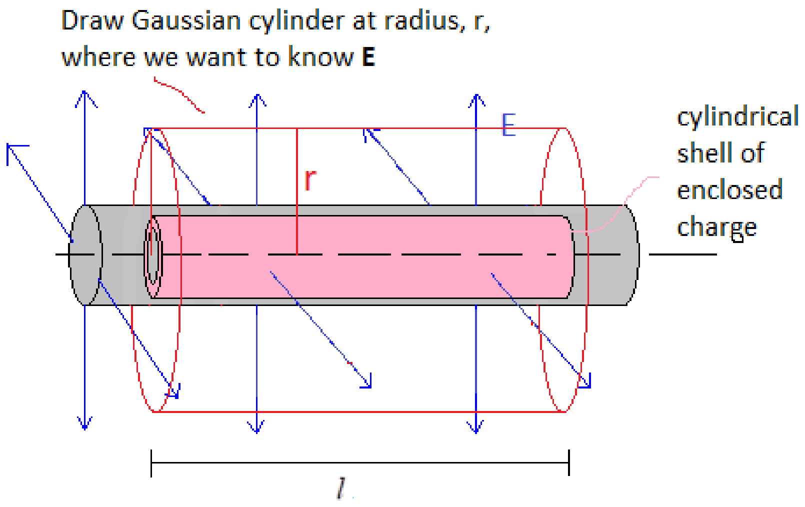
$$\mathbf{E} = \frac{|q_{enc.}|}{4\pi\varepsilon_0 r^2}$$ @ away from/towards a positive/negative sphere

$$q_{enc.} = \int \rho dV_{spherical\ shell} = \int \rho (A_{sphere} dr) = \int \rho 4\pi r^2 dr$$

A.2 Gauss’s Law

Now we’ll use Gauss’s law to get the electric field of a cylindrically symmetric charge distribution





$$\oint \mathbf{E}\cdot d\mathbf{A} = \frac{q_{enc.}}{\varepsilon_0}$$

$$\oint E dA = \frac{q_{enc.}}{\varepsilon_0}$$ because **E** is parallel to **A** everywhere

$$E\oint dA = \frac{q_{enc.}}{\varepsilon_0}$$ can pull E out because it is constant everywhere on the surface

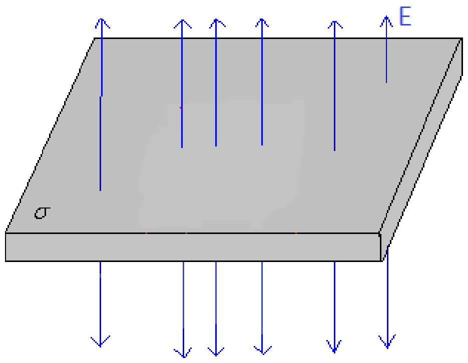
$$E\left[l 2\pi r\right] = \frac{q_{enc.}}{\varepsilon_0}$$ the sum of A over all surfaces just gives the total surface area

$$E = \frac{q_{enc.}/l}{2\pi\varepsilon_0 r} \qquad \lambda_{encl} = \frac{q_{enc.}}{l} = \frac{\int \rho dV_{cylindrical\,shell}}{l} = \frac{\int \rho (l dA_{ring})}{l} = \int \rho dA_{ring} = \int \rho\cdot(2\pi r dr)$$

$$\mathbf{E} = \frac{\left|\lambda_{enc.}\right|}{2\pi\varepsilon_0 r}$$ @ away from/towards a positive/negative cylinder

A.2 Gauss's Law

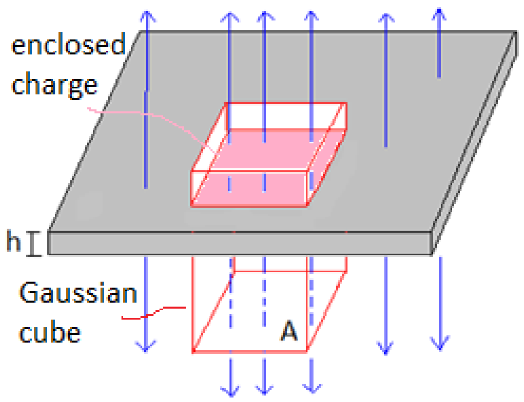
And now the field of a planar symmetric charge distribution



$$\oint \mathbf{E}\cdot\mathbf{A} = \frac{q_{enc}}{\varepsilon_0}$$

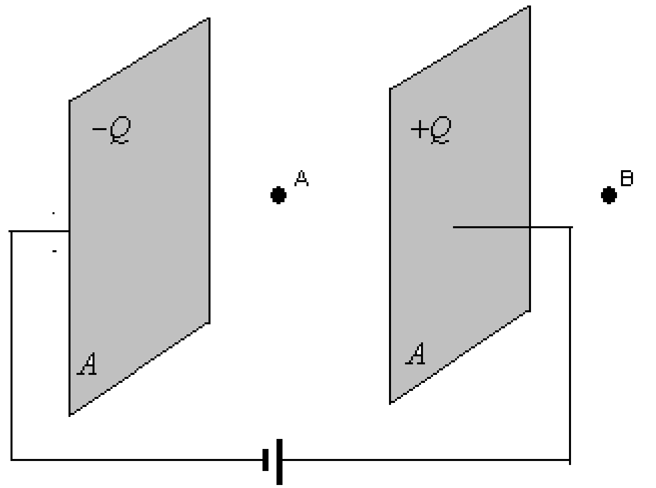
$$EA + EA = \frac{q_{enc.}/A}{\varepsilon_0}$$

$$E = \frac{q_{enc.}/A}{2\varepsilon_0} \qquad \sigma_{enc.} = \frac{q_{enc.}}{A} = \frac{\int \rho dV}{A} = \rho\frac{\int dV}{A} = \rho\frac{hA}{A} = \rho h$$

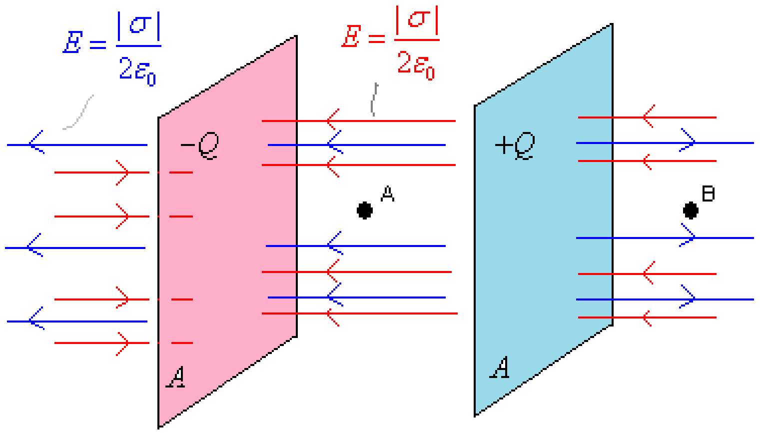


$$\mathbf{E} = \frac{|\sigma_{enc.}|}{2\varepsilon_0}$$ @ away from/towards a positive/negative sheet

A.2 Gauss's law



Consider two parallel plates with area A = 50cm^2 hooked up to a battery which deposits Q = 10mC on each plate. These two plates would constitute a 'capacitor', which are often used to power electrical devices. Anyway, what is the electric field at points A and B?



Can see that fields of two plates cancel out everywhere except in between, where they augment. So, at point B we get E = 0, and at point A we get:

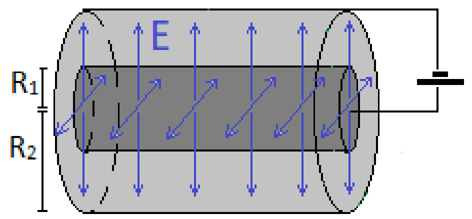
$$\mathbf{E}_{\text{point }A} = 2\cdot\frac{|\sigma|}{2\varepsilon_0}(-\hat{\mathbf{i}})$$

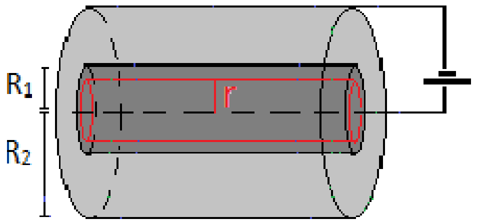
$$= 2\frac{10\times10^{-3}\,\mathrm{C}/50(0.01\mathrm{m})^2}{2(8.85\times10^{-12}\,\mathrm{C^2m^2/N})}(-\hat{\mathbf{i}})$$

$$= -226\,\mathrm{kN}\,\hat{\mathbf{i}}$$

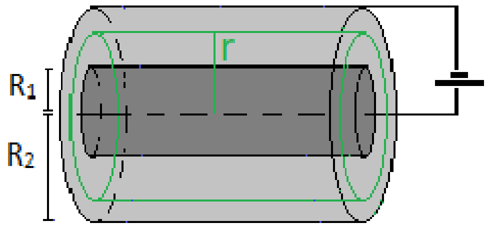
A.2 Gauss's Law

A Geiger counter consists of two concentric, oppositely charged cylinders. A strong electric field, just on the verge of ionizing the air (E = 3kN/C) is generated between the cylinders. When radiation enters the chamber, it will kick an electron off an air molecule, and precipitate an avalanche of current onto the positive terminal, which registers the presence of the radiation. The inner rod is uniformly charged throughout with charge density $\rho(r) = Cr^2$. What is the electric field at all radii (three typical ones are shown: inside inner cylinder, between cylinders, outside cylinders)?

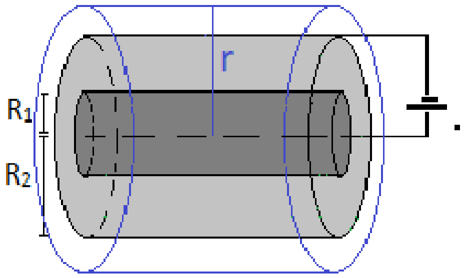




$$E = \frac{\lambda_{enclosed}}{2\pi\varepsilon_0 r} = \frac{\int_0^r \rho dA_{ring}}{2\pi\varepsilon_0 r} = \frac{\int_0^r (Cr^2)\cdot 2\pi r dr}{2\pi\varepsilon_0 r} = \frac{\frac{C}{2}\pi r^4}{2\pi\varepsilon_0 r} = \frac{C}{4\varepsilon_0} r^3$$



$$E = \frac{\lambda_{enclosed}}{2\pi\varepsilon_0 r} = \frac{\int_0^{R_1} \rho dA_{ring}}{2\pi\varepsilon_0 r} = \frac{\int_0^{R_1} (Cr^2)\cdot 2\pi r dr}{2\pi\varepsilon_0 r} = \frac{C}{4\varepsilon_0}\frac{R_1^4}{r}$$



$$E = \frac{\lambda_{enclosed}}{2\pi\varepsilon_0 r} = \frac{0}{2\pi\varepsilon_0 r} = 0$$